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Convergence of subdivision and degree elevation

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This paper presents a short, simple, and general proof showing that the control polygons generated by subdivision and degree elevation converge to the underlying splines, box-splines, or multivariate Bézier polynomials, respectively. The proof is based only on a Taylor expansion. Then the results are carried over to rational curves and surfaces. Finally, an even shorter but as simple proof is presented for the fact that subdivided Bézier polygons converge to the corresponding curve.

1. Introduction

Subdivision and degree elevation are well-known techniques in computer-aided curve and surface design. Very briefly, subdivision and degree elevation mean to represent a curve or surface given as a linear combination of some basis functions with respect to a different but specific set of new basis functions. The process of subdivision and degree elevation can be iterated so as to produce a sequence of so-called control polygons which converges to the underlying curve or surface.

These techniques are rather useful in a number of applications such as curve and surface evaluations [1] and display [3,12], surface/surface intersections [12], and they can also facilitate simple proofs, e.g. about convexity [19].

The convergence property is well-known and established in a number of publications by different authors: Farin [8–10], Micchelli et al. [14], Dahmen [6], Cohen et al. [4], Lane et al. [12], Prautzsch [16,17], Dahmen et al. [5]. Another proof of the convergence of subdivision which is quite well known in the CAGD community, though not published, makes use of the polar form of polynomials [13,20].

One also has similar results for more general subdivision schemes, e.g. [2,18], and general corner cutting schemes [7,11].

The references mentioned above use about seven different ideas to prove convergence. However, only two ideas are useful to find the rates of convergence, but nowhere can one find the convergence constants. Moreover, none of these proof techniques seem to be applicable to all the algorithms which subdivide or degree elevate the Bézier, B-, or box spline representation of a curve or surface. In particular, there is no (correct) proof of the convergence of degree elevated spline control polygons in these references, cf. section 5.

In this paper, we will present a general and also rather simple proof technique. The only prerequisite is a knowledge of Taylor's expansion. Moreover, we obtain explicit constants. Then we show how the convergence results for integral curves and surfaces can be carried over to the rational case. Finally, we present a second, even shorter proof of the fact that subdivided Bézier polygons converge.

Remark 1

It should be noted that *nowhere* in this paper do we consider the process of computing a degree raised or subdivided representation of splines explicitly. Rather, we consider a fixed spline of degree n in some mth degree representation and estimate the deviation of the corresponding control points depending on m and the knot spacing.

The final estimates therefore imply simultaneously that sequences of control polygons produced either by subdivision or degree elevation for some fixed spline converge at a certain rate.

2. A simple general idea to prove convergence

In this section, we explain a general method to prove convergence under subdivision and degree elevation. This is done best by a simple example, namely for Bézier curves over the unit interval [0, 1]. An arbitrarily small interval is considered in the next section.

Let $B_i^m(x)$, $i = 0, \ldots, m$, be the Bernstein polynomials of degree m and let

$$p(x) = \sum_{i=0}^{m} b_i B_i^m(x)$$

be some polynomial of degree $n \leq m$ in its mth degree Bernstein–Bézier representation. By induction over n, we will prove that

$$\max_i \left| b_i - p\left(\frac{i}{m}\right) \right| = O\left(\frac{1}{m}\right). \tag{2.1}$$

For $n \leq 1$, one has $b_i = p(i/m)$ since the Bernstein–Bézier representation has linear precision. Next, we assume that (2.1) holds for polynomials of degree $n-1$. On writing

$$b_i = b_0 + \sum_{k=0}^{i-1} \frac{1}{m}[m\Delta b_k]$$

and applying (2.1) to $p'(x) = \sum_{k=0}^{m-1} m\Delta b_k B_k^{m-1}(x)$, we obtain

$$b_i = p(0) + \sum_{k=0}^{i-1} \frac{1}{m}\left[p'\left(\frac{k}{m-1}\right) + O\left(\frac{1}{m}\right)\right].$$

Let $a = k/m$, $c = k/(m-1)$, and $b = (k+1)/m$. Then $a \le c \le b$ and by Taylor's theorem there are numbers ξ and η in $[a, b]$ such that

$$\begin{aligned} p(a) &= p(c) + (a-c)p'(c) + \frac{(a-c)^2}{2} p''(\xi), \\ p(b) &= p(c) + (b-c)p'(c) + \frac{(b-c)^2}{2} p''(\eta), \end{aligned} \tag{2.2}$$

which gives

$$(b-a)p'(c) = p(b) - p(a) + \tfrac{1}{2}(b-a)^2 E, \quad |E| \le \max_{[a,b]} |p''|. \tag{2.3}$$

Thus, one has

$$\begin{aligned} b_i &= p(0) + \sum_{k=0}^{i-1} \left[p\left(\frac{k+1}{m}\right) - p\left(\frac{k}{m}\right) + O\left(\frac{1}{m^2}\right) \right] \\ &= p\left(\frac{i}{m}\right) + O\left(\frac{1}{m}\right), \end{aligned}$$

which is (2.1).

Note that $p^{(n+1)} \equiv 0$. Hence, a more careful analysis establishes

$$\begin{aligned} \left| b_k - p\left(\frac{k}{m}\right) \right| \le \frac{1}{2m} \max_{[0,1]} |p''| + \frac{1}{2(m-1)} \max_{[0,1]} |p'''| \\ + \ldots + \frac{1}{2(m-n+2)} \max_{[0,1]} \left| p^{(n)} \right|. \end{aligned}$$

Notice that the constants in the above estimate are not optimal. Moreover, since p is a polynomial, the remaining term in Taylor's theorem also is a polynomial. Therefore, one can write $b_k - p(k/m)$ as a rational polynomial of degree $2n-1$ in m.

3. Bézier simplices

The proof of the previous section also applies to multivariate polynomials in Bézier representation. Let $\boldsymbol{i} = (i_1, \ldots, i_d)$, $\boldsymbol{u} = (u_1, \ldots, u_d)$, and $i_0 = m - i_1 - \ldots - i_d$; then the Bernstein polynomials of degree m are

$$B_{\boldsymbol{i}}^m(\boldsymbol{u}) = \frac{(1 - u_1 - \ldots - u_d)^{i_0} u_1^{i_1} \ldots u_d^{i_d}}{i_0! \ \ldots \ i_d!}.$$

Notice that here $B_{\boldsymbol{i}}$ is not represented in the usual symmetric form. Further, let

$$I_m = \{\boldsymbol{i} \in \{0, 1, \ldots, m\}^d : i_0 \ge 0\}$$

and consider some polynomial p of degree $n \le m$ in its mth degree Bézier representation

$$p(u) = \sum_{i \in I_m} b_i B_i^m\left(\frac{u-a}{h}\right).$$

As in the previous section, one can show by induction on n that

$$\max_{i \in I_m} |b_i - p(a + hi/m)| \le E_n, \tag{3.1}$$

where E_n is recursively defined by

$$\begin{aligned} E_{n-k} &= \frac{h^2}{(m-k)} D^{k+2} + hE_{n-k-1}, \\ E_1 &= 0, \\ D^k &= \sum_{q \in Q} \max\{|q(u)| : \mathbf{0} \le u, u_1 + \ldots + u_d \le 1\}, \\ Q &= \{\text{all partial derivatives of } p \text{ of order } k\}. \end{aligned}$$

Note that $E_n = O(h^2/m)$.

For $n = 1$, one has $b_i = p(a + hi/m)$. Then for the induction, we assume that (3.1) holds for all polynomials of degree $n - 1$ and derive (3.1) for p of degree n. Let $i \in I_m$ and $a = i_1 + \ldots + i_d$. Then consider a sequence $j_0 = \mathbf{0}, j_1, \ldots, j_a = i$, where all differences Δj_k equal some unit vector $(0, \ldots, 0, 1, 0, \ldots, 0)$. The vertices marked in fig. 1 form such a sequence, where $m = 7$ and $d = 2$.

On using the abbreviation $a_k = b_{j_k}$, one has

$$b_i = a_0 + \sum_{k=0}^{a-1} \frac{h}{m}\left[\frac{m}{h}\Delta a_k\right].$$

Note that $m\Delta a_k/h$ represents the control point c_{j_k} of the directional derivative

$$p_k'(u) = \frac{\mathrm{d}}{\mathrm{d}t} p(u + t\Delta j_k)|_{t=0} = \sum_{i \in I_{m-1}} c_i B_i^{m-1}(u).$$

Therefore, if one applies the induction hypothesis to these derivatives one obtains

$$b_i = p(a) + \sum_{k=0}^{a-1} \left[\frac{h}{m} p_k'\left(a + h\frac{j_k}{m-1}\right) + \frac{h}{m} F\right],$$

where

$$|F| \le E_{n-1}.$$

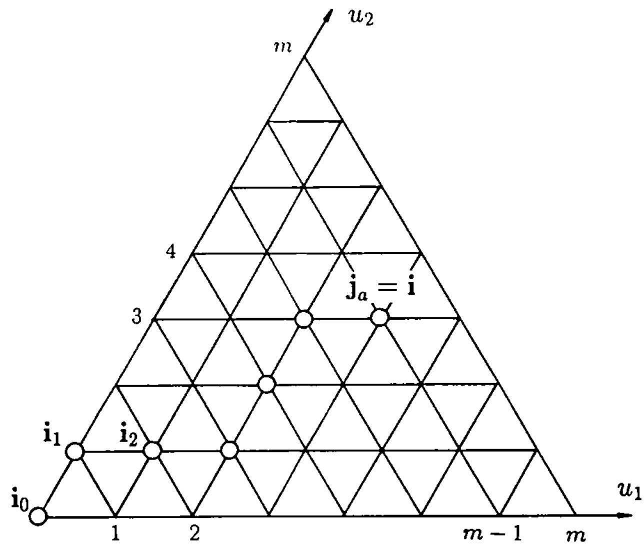

Figure 1. A sequence (j_k).

Notice that $j_k/(m-1)$ lies in the interval $[j_k, j_k + (1, 1, \dots, 1)]/m$. Thus, by successive applications of Taylor's theorem, we obtain

$$b_l = p(a) + \sum_{k=0}^{a-1}\left[p\left(a + h\,\frac{j_{k+1}}{m}\right) - p\left(a + h\,\frac{j_k}{m}\right) + G + \frac{h}{m}F\right],$$

where

$$|G| \le \frac{h^2}{m^2}D^2,$$

which proves (3.1).

4. Splines

The proof of section 2 can also be used to show convergence under degree elevation and subdivision for splines. Let $(u_i)_{i\in Z}$ be a non-decreasing sequence of knots such that $u_i < u_{i+m+1}$ and let B_i^m be the B-spline of order $m+1$ defined by the knots $u_i, \dots, u_{i+m+1}$. Then consider the spline

$$s(x) = \sum_{i\in Z} b_i B_i^m(x)$$

of some degree $n \le m$. We assume that this representation of s is generated by degree elevation, see e.g. [15], from the nth degree representation. Hence, each knot u_i occurs with multiplicity $m - n + 1$ at least.

Let

$$\xi_i = \frac{1}{m}(u_{i+1} + \ldots + u_{i+m});$$

then one can prove by induction over n that

$$\max_{0 \le i \le m} |b_i - s(\xi_i)| \le E_n, \tag{4.1}$$

where E_n is recursively defined by

$$E_{n-k} = \frac{((n-k)h)^2}{2(m-k)} \max_{1 \le i \le 2m-1} \max_{u_i \le x \le u_{i+1}} |s^{(k+2)}(x)| + (n-k)hE_{n-k-1},$$
$$E_1 = 0,$$

and $h = \max_{k=1}^{2m-1} |\Delta u_k|$. For $n \le 1$, it is well known that $b_i = s(\xi_i)$. For the induction, we assume that (4.1) holds for splines of order $n-1$ and derive (4.1) for s of order n. Considering $|b_i - s(\xi_i)|$ we can assume, without loss of generality, that u_{i+1} is an m-fold knot since, otherwise, one could insert knots at u_{i+1} such that b_i and ξ_i remain unchanged. Thus, there is some $a \le i+1$ such that

$$\xi_a = u_{a+1} = \ldots = u_{i+1} = \ldots = u_{a+m}.$$

This means $b_a = s(\xi_a)$. Again, we write

$$b_i = s(\xi_a) + \sum_{k=a+1}^{i} \frac{1}{\gamma_k}[\gamma_k \nabla b_k], \quad \gamma_k = \frac{m}{u_{k+m} - u_m},$$

and observe that $\gamma_k \nabla b_k$ represents the control point c_k of $s'(x) = \sum_{i=1}^{m} c_i B_i^{m-1}(x)$. Thus, on applying (4.1) to $s'(x)$, we obtain

$$b_i = s(\xi_a) + \sum_{k=a+1}^{i} \frac{1}{\gamma_k}[s'(\eta_k) + F],$$

where $\eta_k = (1/(m-1))(u_{k+1} + \ldots + u_{k+m-1})$ and $|F| \le E_{n-1}$. Since each knot u_i occurs with multiplicity $\ge m-n+1$, one has $u_{k+m} - u_k \le nh$. Further, notice that $\eta_k \in [\xi_{k-1}, \xi_k]$. Hence, we can proceed as in (2.3) and obtain

$$b_i = s(\xi_a) + \sum_{k+a+1}^{i} [s(\xi_k) - s(\xi_{k-1}) + \frac{1}{\gamma_k} F + G],$$

where

$$|G| \le \frac{1}{2}\left(\frac{n}{m}h\right)^2 \max_{x \in [u_1, u_{2m}]} |s''(x)|.$$

Computing the above sum establishes (4.1).

Remark 2

In order to apply Taylor's theorem, we need $s \in C^2(\xi_{k-1}, \xi_k)$, $k = a+1, \ldots, i$. This is the case since there is no $(m-1)$-fold knot in the open interval $(\xi_a, \xi_i) \subset (u_{i+1}, u_{i+m})$.

5. A counter example

Before we continue with box splines, we briefly recall the general proof technique used in most afore-mentioned references and show its limitation.

Consider the linear space spanned by the basis splines $B_0^m, \ldots, B_m^m$. Since norms are equivalent in this space, there is a constant K_m such that for all splines

$$s(x) = \sum_{i=0}^{m} b_i B_i^m(x)$$

in this space, one has the *stability property*

$$\max_{0 \le i \le m} |b_i| \le K_m \max_{x \in [u_0, u_{2m+1}]} |s(x)|.$$

In order to estimate the deviation of the control points b_i from $s(x)$, one can employ the quasi-interpolant

$$Qs = \sum s(\xi_i) B_i^m.$$

Then, on using the stability property, one obtains

$$\max_{0 \le i \le m} |b_i - s(\xi_i)| \le K_m \max_{[u_0, u_{2m+1}]} |s - Qs|.$$

The more difficult part is to estimate K_m and $s - Qs$. In [21], one finds

$$K_m \approx 2^m \quad \text{and} \quad \max |s - Qs| = O\left(\frac{h^2}{m}\right),$$

where h is the maximum difference between successive knots. Obviously, the above estimates alone do not prove convergence under degree elevation.

However, if the B_i^m are Bernstein polynomials, there is a solution [14]. In this case, s and Qs are polynomials of some degree $n \le m$. Hence, $b_0^m - s(\xi_0^m), \ldots, b_m^m - s(\xi_m^m)$ is a degree elevated control polygon of $s - Qs$. It lies in the convex hull of the non-degree elevated control polygon of $s - Qs$, say $c_0, \ldots, c_n$. Thus, one has

$$\max |b_i^m - s(\xi_i)| \le \max |c_i| \le K_n \max |s - Qs| \le O\left(\frac{1}{m}\right).$$

As one may expect, this solution does not apply to splines in general. For example, let (u_i) be a sequence of $(m-1)$-fold knots such that $u_0 = u_1 = -1$, $u_2 = \ldots = u_m = 0$, $u_{m+1} = \ldots = u_{2m-1} = 1$, and $u_{2m} = u_{2m+1} = 2$, and let

$$s(x) = \sum_{i \in \mathbb{Z}} b_i B_i^m(x)$$

be the mth degree representation of the parabola $s(x) = x^2$. Then, over [0, 1] one has

$$Qs = \frac{1}{m^2}\left(B_0^m + 1^2 B_i^m + 2^2 B_2^m + \ldots + (m-1)^2 B_{m-1}^m + (m+1)^2 B_m^m\right),$$

which is of proper polynomial degree $m-1$ or m.

The proof given in [4] compares in essence to the lines above, but misses the problem addressed by our example. It appears to us that until now there has not been any proof that degree elevation of splines converges linearly.

6. Box splines

After some additional provisions, the proof of section 2 also applies to the subdivision algorithm of box splines.

Let $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ denote a family of not necessarily distinct vectors in $\mathbb{Z}^s$ and also the matrix whose columns are $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$. Furthermore, assume that for each linearly independent subset Y of X, one has $\det Y = \pm 1$. Notice that this assumption implies $X \subset \{-1, 0, 1\}^s$. Then the box spline $B(\boldsymbol{x} \mid X)$ is defined by the requirement that

$$\int_{\mathbb{R}^s} f(\boldsymbol{x}) B(\boldsymbol{x} \mid X) \mathrm{d}\boldsymbol{x} = \int_{[0,1]^n} f(X\boldsymbol{y}) \mathrm{d}\boldsymbol{y}$$

holds for all continuous functions f on $\mathbb{R}^s$. Now consider a spline

$$s(\boldsymbol{x}) = \sum_{\boldsymbol{i} \in \mathbb{Z}^s} b_{\boldsymbol{i}} B(\boldsymbol{x} - \boldsymbol{i} \mid X).$$

Subdividing $s(\boldsymbol{x})$ means to represent $s(\boldsymbol{x})$ over the finer grid $h\mathbb{Z}^s$, $h^{-1} \in \mathbb{N}$, by translates of the scaled box spline $B(\boldsymbol{x}/h \mid X)$, i.e. subdivision means to produce control points $c_{\boldsymbol{i}}$ such that

$$s(\boldsymbol{x}) = \sum_{\boldsymbol{i} \in \mathbb{Z}^s} c_{\boldsymbol{i}} B(\boldsymbol{x}/h - \boldsymbol{i} \mid X).$$

Let $Y = \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_r\} \subset X$ be such that $B(\boldsymbol{x} \mid X \setminus Y)$ is continuous and piecewise linear (in every direction). Then, on using the idea of section 3, we can prove by induction on r that

$$\sup_{\boldsymbol{i} \in \mathbb{Z}^s} |s(\boldsymbol{\xi}_{\boldsymbol{i}}) - c_{\boldsymbol{i}}| \leq E_n, \tag{6.1}$$

where $\boldsymbol{\xi}_{\boldsymbol{i}} = h\boldsymbol{i} + (h/2)(\boldsymbol{x}_1 + \ldots + \boldsymbol{x}_n)$ and

$$E_{n-k} = \frac{2(n-k)h^2}{3} \sup_{j \in Z^s} 2^k |b_j| + (n-k)E_{n-k-1}, \quad k = 0, \dots, r-1,$$
$$E_{n-r} = 0.$$

Recall that

$$s(\xi_i) = c_i$$

if s is piecewise linear in each direction $x_i \in X$, i.e. $Y = \emptyset$.

For the induction, we assume $Y_r \neq \emptyset$ and that (6.1) holds if X did not contain direction y_1. Because of possible index shifts, it suffices to consider $c_i - s(\xi_i)$ for $i = (n/h, 0, \dots, 0)$. We may then set the following control points to zero:

$$b_j = 0 \quad \text{for } j[1] \leq 0.$$

This change does not affect $c_i - s(\xi_i)$, nor does it increase $\sup_{Z^s} |b_j|$. Then we have $c_0 = s(\xi_0) = 0$ for $h \leq 2/n$. On using the abbreviations $c_k = c_{(k,0,\dots,0)}$ and $\xi_k = \xi_{(k,0,\dots 0)}$, we obtain

$$c_i = c_0 + \sum_{k=1}^{n/h} \nabla c_k.$$

Note that $\nabla c_k / h$ represents the control point $d_{(k,0,\dots,0)}$ of the directional derivative of s with respect to the first unit vector

$$Ds(x) = \frac{\mathrm{d}}{\mathrm{d}t} s(x + te_1)|_{t=0} = \sum_{i \in Z^s} d_i B(x/h - i \,|\, X \setminus \{e_1\}).$$

Thus, on using (6.1) for Ds, we obtain

$$c_i = c_0 + \sum_{k=1}^{n/h} (hDs(\eta_k) + hF),$$

where $\eta_k = (\xi_{k-1} + \xi_k)/2$ and $|F| \leq E_{n-1}$. Thus, we can conclude by means of Taylor's theorem as in (2.3) that

$$c_i = \sum_{k=1}^{n/h} (\nabla s(\xi_k) + G + hF),$$

where

$$|G| \leq \frac{h^3}{12} \sup_{x \in \mathbb{R}^s} |D^3 s(x)| \leq \frac{2h^3}{3} \sup_{k \in Z^s} |b_k|.$$

Note that η_k is the midpoint of ξ_{k-1} and ξ_k. If in (2.2) $c = (a+b)/2$, the second-order terms in (2.2) agree and (2.3) is a third-order approximation of $(b-a)p'(c)$. Therefore, the above estimate involves a third-order derivative.

Remark 3

Note that Ds exists continuously, but D^2s may be discontinuous. If $Ds(\eta_k)$ is discontinuous, then it is discontinuous only in $\mathbb{Z}$. These discontinuities do not affect the above estimate.

7. Rational splines

The convergence results also hold for rational curves and surfaces, i.e. subdivision converges quadratically and degree elevation linearly. This can be seen as follows. Let

$$r(x) = \frac{\sum \beta_i b_i B_i(x)}{\sum \beta_i B_i(x)} = \frac{q(x)}{\beta(x)}$$

be a rational curve or surface where the B_i are either uni- or multivariate Bernstein polynomials, B- or box splines. Then we can apply (2.1), (3.1), (4.1), or (6.1) to both the numerator and denominator. The respective convergence results are of the form

$$\max_i |\beta_i b_i - q(\xi_i)| \le A$$

and

$$\max_i |\beta_i - \beta(\xi_i)| \le C.$$

We will only consider positive weights β_i. Therefore, without loss of generality, we can assume that all $\beta_i \ge 1$. With this provision, it is quite simple to show that the control points b_i converge to the function values $r(\xi_i)$, namely

$$\begin{aligned} |b_i - r(\xi_i)| &\le \beta_i |b_i - r(\xi_i)| \\ &\le |\beta_i b_i - q(\xi_i)| + |r(\xi_i)|\,|\beta(\xi_i) - \beta_i| \\ &\le A + \max_i |r(\xi_i)| C. \end{aligned}$$

Note that $r(x)$ is bounded over a bounded interval since $\beta(x)$ is positive.

8. Another simple proof

Finally, we present another quite different proof for the convergence of Bézier polygons under subdivision. Let

$$p(x) = \sum_{i=0}^{n} b_i(h) B_i^n\left(\frac{x-a}{h}\right)$$

be a polynomial with its Bézier representation over the interval $[a, a+h]$. Then it follows directly from de Casteljau's algorithm that for all $j = 0, 1, \ldots, n$

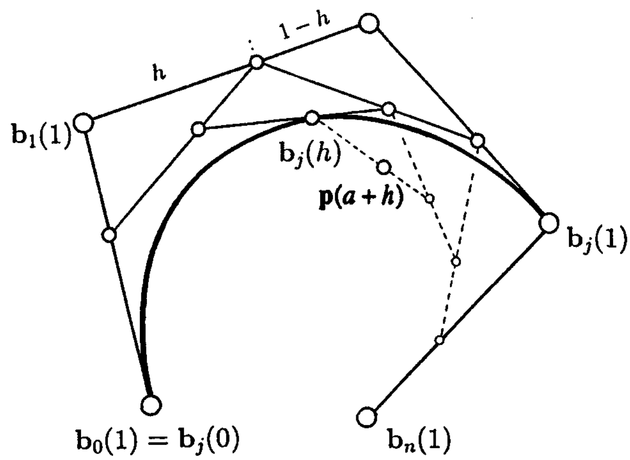

Figure 2. De Casteljau's construction.

$$b_j(h) = \sum_{i=0}^{j} b_i(1) B_i^j(h),$$

i.e. $b_j(h)$ traces out the curve with Bézier points $b_0(1), \ldots, b_j(1)$ for varying h, as illustrated in fig. 2. Note that $b_j(0) = p(a)$ and $b_j'(0) = (j/n)p'(a)$. Thus, by Taylor's theorem it follows that

$$b_j(h) = p\left(a + \frac{j}{n}h\right) + O(h^2).$$

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